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## ABSTRACT

Numerical integration methods for solution of the system of differential equations found in ballistic rocket trajectory programs are discussed.

The general discussion entails the explicit formulas of Runge-Kutta and predictor-corrector methods and their errors, and a brief description of other methods that could be employed.

CONTENTS

## INTRODUCTION

In a ballistic rocket trajectory simulation program the system of differential equations used to describe the ballistic model is a highly complex system. In particular the six-degree of freedom model used by the Atmospheric Sciences Office at White Sands Missile Range, consists of a system of twenty-one 2nd order ordinary differential equations which are to be solved for the ballistic rocket's components of acceleration, velocity, and position at discrete time intervals. The most feasible method for solving the system is to program it for a computer and solve by some numerical integration procedure.

This report will present several numerical integration schemes which are currently being used or are feasible for use in a ballistic rocket simulation program.*

## RUNGE-KUTTA METHODS

### Formulas

Consider the system of first order differential equations

$$y_i' = \frac{dy_i}{dx} = f_i(x, y_1(x), y_2(x), \ldots, y_N(x)) \tag{1}$$

satisfying the initial conditions

$$y_i(x_o) = y_{io}. \tag{2}$$

We desire the values of $y_i(x_o + h)$, $i = 1, 2, \ldots, N$, where h is the increment in the independent variable.

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The Runge-Kutta method is an algorithm designed to approximate the Taylor series solution

$$y_i(x_o + h) = y_{io} + h\, y_i^{(1)}(x_o) + \frac{h^2}{2}\, y_i^{(2)}(x_o) + \frac{h^3}{6}\, y_i^{(3)}(x_o) + \ldots \tag{3}$$

of (1), where $y_i^{(k)}(x_o)$ denotes the kth derivative and i = 1, 2, ..., N. However, in deriving the defining equations of the algorithm, the use of the appropriate assumptions makes it unnecessary to evaluate and define the derivatives of higher order.

The general equations defining a fourth order Runge-Kutta method are given by

$$\begin{aligned} K_1 &= hf(x_o, y_o) \\ K_2 &= hf(x_o + \alpha h, y_o + \beta K_1) \\ K_3 &= hf(x_o + \alpha_1 h, y_o + \beta_1 K_1 + \gamma_1 K_2) \\ K_4 &= hf(x_o + \alpha_2 h, y_o + \beta_2 K_1 + \gamma_2 K_2 + \delta_2 K_3) \\ \overline{K} &= \overline{y}(x_o + h) - y_o = \mu_1 K_1 + \mu_2 K_2 + \mu_3 K_3 + \mu_4 K_4 \end{aligned} \tag{4}$$

where the bar denotes an approximate value.

The thirteen parameters involved are found by expanding $\overline{K}$ about the point $(x_o, y_o)$ so as to agree with the Taylor series expansion (3) up to and including terms of order $h^4$; thus, the fourth order method. Initially, the solution will result in eight equations in thirteen unknowns. However, under the appropriate assumptions, the system will resolve into a system of eight equations in nine unknowns, hence a one parameter family.

One solution is the classical formulas of Runge, given by,

$$
\begin{aligned}
K_1 &= hf(x_o, y_o) \\
K_2 &= hf(x_o + \frac{h}{2}, y_o + \frac{K_1}{2}) \\
K_3 &= hf(x_o + \frac{h}{2}, y_o + \frac{K_2}{2}) \\
K_4 &= hf(x_o + h, y_o + K_3) \\
\overline{K} &= \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)
\end{aligned}
\tag{5}
$$

One will notice that when $y' = f(x)$, Runge's formulas reduce to Simpson's rule:

$$
\overline{K} = \int_{x_o}^{x_o+h} f(x)dx = \frac{h}{6}\left[f(x_o) + 4f\left(x_o + \frac{h}{2}\right) + f(x_o + h)\right] \tag{6}
$$

Another solution to the one parameter family is the classical formulas of Kutta given by

$$
\begin{aligned}
K_1 &= hf(x_o, y_o) \\
K_2 &= hf(x_o + \frac{h}{3}, y_o + \frac{K_1}{3}) \\
K_3 &= hf(x_o + \frac{2h}{3}, y_o - \frac{K_1}{3} + \frac{K_2}{3}) \\
K_4 &= hf(x_o + h, y_o + K_1 - K_2 + K_3) \\
\overline{K} &= \frac{1}{8}(K_1 + 3K_2 + 3K_3 + K_4)
\end{aligned}
\tag{7}
$$

To apply the Runge-Kutta method using a high speed digital computer, Gill [8] developed a calculation procedure which

(a) requires a minimum number of storage registers,
(b) gives a high accuracy,
(c) requires comparatively few instructions.

Gill accomplished the above by defining an auxiliary set of parameters in the initial solution set, thereby obtaining the following solution;

$$K_1 = hf(x_o, y_o) \qquad y_1 = y_o + \frac{1}{2}(K_1 - 2Q_o)$$

$$K_2 = hf(x_o + \frac{h}{2}, y_1) \qquad y_2 = y_1 + (1 - \sqrt{\frac{1}{2}})\,(K_2 - Q_1)$$

$$K_3 = hf(x_o + \frac{h}{2}, y_2) \qquad y_3 = y_2 + (1 + \sqrt{\frac{1}{2}})\,(K_3 - Q_2)$$

$$K_4 = hf(x_o + h, y_3) \qquad y_4 = y_o + \bar{K} = \bar{y}\,(x_o + h)$$

$$\bar{K} = \frac{1}{6}K_1 + \frac{1}{3}(1 - \sqrt{\frac{1}{2}})\,K_2 + \frac{1}{3}(1 + \sqrt{\frac{1}{2}})\,K_3 + \frac{1}{6}K_4 \qquad (8)$$

$$Q_1 = Q_o + 3[1/2(K_1 - 2Q_o)] - \frac{1}{2}K_1$$

$$Q_2 = Q_1 + 3[(1 - \sqrt{\frac{1}{2}})\,(K_2 - Q_1)] - (1 - \sqrt{\frac{1}{2}})\,K_2$$

$$Q_3 = Q_2 + 3[(1 + \sqrt{\frac{1}{2}})\,(K_3 - Q_2)] - (1 + \sqrt{\frac{1}{2}})\,K_3$$

$$Q_4 = Q_3 + 3[\frac{1}{6}(K_4 - 2Q_3)] - \frac{1}{2}K_4$$

where the $Q_i$ are the auxiliary parameters. Initially $Q_o$ is zero; to compensate for the round-off error in $y_4$, $Q_4$ is used as the $Q_o$ in the next step. For a complete discussion of the algorithm and a flow chart, refer to [16].

## Error

To obtain a general expression for truncation error per step, the functions $\bar{K} = \bar{y}(x_o + h) - y_o$ and $K = y(x_o + h) - y_o$ are expanded through terms of $h^5$, thus giving the per-step truncation error as

$$E = K - \bar{K} = \frac{h^5}{1440}\left[-\frac{T^4}{2} - 3(S^2T - ST^2 + f^2y\right.$$

$$\left. + 3f_{yy}P + 2f_yST - 4f^3_yT) + 2f_yT^3\right] + \ldots \qquad (9)$$

where $T^K = D^K f$, $S^K = D^K fy$, $P = (Df)^2$.

Lotkin [12] determined a bound for E which is given by

$$|E| \leq \frac{73}{720} ML^4h^5 \qquad (10)$$

where in R*, a region containing $(x_o, y_o)$,

$$|f(x, y)| \leq M,$$

and

$$\left|\frac{\partial^{i+j}f}{\partial x^i\, \partial y^j}\right| \leq \frac{L^{i+j}}{M^{j-1}}, \qquad (11)$$

where the positive constants M and L, are independent of (x, y), for $i + j \leq 4$.

A numerical integration method is called stable if at the nth step, the total error, which is both truncation and round-off error, is at a minimum, thereby forcing the approximate solution to tend to the true solution.

Hence, as for the stability of the Runge-Kutta methods, Carr [3] proved a theorem, of which the essential parts are stated below:

Let $f_y$ be continuous, negative and bounded from above and below in some region R of the (x, y) plane, i.e., $M_2 > M_1 > 0$, $f_y \in (-M_2, -M_1)$; further, let E be the absolute value of the maximum error introduced at each step, and D* be a region in which the solution of the difference equation tends to the y-boundary of D no closer than $Qh + |\epsilon_i|$, where

$$Q \geq \max_{x,y \in D} |f(x, y)| \geq \max \left[\left|\frac{K_1}{2}\right|, \left|\frac{K_2}{2}\right|, |K_3|\right] \tag{12}$$

and $\epsilon_i$ is the error at the ith step. Then, the Kutta fourth order method has a bound on the total error in the ith step, where this bound is given by

$$|\epsilon_i| \leq \frac{2E}{hM_1} \tag{13}$$

in some region D*, and where the step-size, h, is given by

$$h < \min \left(\frac{M_1}{M_2^2}, \frac{4M_1^3}{M_2^4}\right). \tag{14}$$

If, $f_y \geq 0$ (the unstable case), but bounded throughout D, $0 \leq f_y \leq M$, then the propagated error, $\lambda_{i+1}$ in D* at the (i + 1) st step is given by

$$|\lambda_{i+1}| \leq |\epsilon_i| \, e^{hM} \tag{15}$$

and

$$|\varepsilon_i| \leq E\left(\frac{e^{ihM} - 1}{e^{hM} - 1}\right)$$

and E and h are as given before.

Statements (13), (14), (15) relate the step-size and the propagated error. Thus, an h can be found that will make the propagated error less than a certain bound, if this bound is the known bound on the partial derivative. Algorithms for finding such step-sizes can be found in Carr [3].

Although various bounds on the truncation error are known, such as the one given above, their usage in a computer program is usually not feasible. Thus, some practical scheme for estimation of this error must be devised.

One such scheme was devised by Richardson [5]. This scheme is based on numerically integrating with step-sizes of h and $\frac{h}{2}$ and comparing the results using these step-sizes. A variation of this which we use is the following:

Let Y be the true value of y at $x_o+h$, $y^{(1)}$ the value obtained at $x_o+h$ using $h_1 = h$, $y^{(2)}$ the value obtained at $x_o+h$ using $h_2 = \frac{h}{2}$. Then for small h, (16)

$$Y-y^{(2)} \doteq \frac{y^{(2)} - y^{(1)}}{y^{(2)}} .$$

The scheme in (16) can be used to check the validity of the answer for the purpose of halving and doubling the step-size h.

## Analysis

By virtue of the Runge-Kutta methods requiring only information from one previous step, the method has desirable stability characteristics and ease of halving or doubling the step-size h.

It has been shown, however, by Blum [2] and Fyfe [6], that through proper redefinition of the solution set of parameters, most other versions of the fourth order Runge-Kutta method can avail themselves of the reduced storage of the Gill algorithm.

If the first derivative of the functions are very involved, then several evaluations of these first derivatives may be somewhat time consuming and thus cause the method to be uneconomical at times. Another distinct shortcoming of the method is that of the error. Neither truncation error, nor its estimate, is obtained in the calculation procedure, thus necessitating the approximate comparisons described previously.

Finally, the Runge-Kutta methods can be used as a starting procedure for other methods, such as multistep methods.

## MULTISTEP METHODS

### Standard Predictor-Corrector Methods

A multistep method is a method in which the calculation of the $y_i$'s at the (M+1) st step depends on knowledge of the $y_i$'s and the $f_i$'s at the Mth, (M-1) st, (M-2)nd, etc. steps. A method is called an m-step method if only m-steps of previous information are required. In the following discussion only four-step methods will be considered since methods with fewer back values can be extracted from the discussion below.

The multistep method with which we are concerned here is known as the predictor-corrector method. This method requires a formula for finding a first estimate of each $y_i$ (hence, the predictor); and then evaluating each function $f_i$ we substitute this into a formula which will adjust the value obtained from the predictor, hence the corrector. By a standard method we mean one in which the same step-size is used in all the equations in integrating for each $y_i$.

First of all, rewrite the system of differential equations in (1) as

$$y_i' = f_i(x, y_1, y_2, \ldots, y_N) \tag{17}$$

with the initial conditions

$$y_i(x_o) = y_i(o) \tag{18}$$

We desire the values $y_i(x_o + h)$, $i = 1, 2, \ldots N$ where h is again the increment in the independent variable.

Next let us clarify some notation which we will be using.

$$y_i(M) = y_i\,(x_o + Mh) \qquad i = 1, 2, \ldots, N$$

$$P_i(M+1) = y_i(x_o + (M+1)h) \qquad i = 1, 2, \ldots, N \tag{19}$$

$$y_i'(M) = y_i'(x_o+Mh) = f_i(x_o+Mh, y_i(M), y_2(M), \ldots, y_N(M))$$

$$\bar{y}_i'(M+1) = \bar{y}_i'(x_o+(M+1)h) = f_i(x_o+(M+1)h, P_1(M+1), \ldots, P_N(M+1)$$

$$C_i(M+1) = y_i(x_o+(M+1)h) \qquad i = 1, 2, \ldots, N$$

By using (17), the defining equations for the standard predictor-corrector algorithm are given by

$$P_i(M+1) = a_1y_i(M) + b_1y_i(M-1) + C_1y_i(M-2)$$

$$+d_1y_i(M-3)+h[e_1y_i'(M) + f_1y_i'(M-1) \tag{20}$$

$$+ g_1y_i'(M-2) + K_1y_i'(M-3)], \; i = 1,2, \ldots, N$$

$$C_i(M+1) = a_2y_i(M) + b_2y_i(M-1) + C_2y_i(M-2)$$

$$+ h[d_2y_i'(M+1) + e_2y_i'(M) + f_2y_i'(M-1) \tag{21}$$

$$+ g_2y_i'(M-2)], \; i =1, 2, \ldots, N$$

where (20) is the predictor, (21) the corrector, and the coefficients are to be determined.

Using the standard technique of [10] for solution of the constant coefficients, the solution set is:

$$a_1 = 9 - d_1 - 3e_1 + 3K_1 \qquad a_2 = 9 - 15d_2 - 3e_2$$

$$b_1 = 9 - 9d_1 + 24K_1 \qquad b_2 = 9 - 24d_2$$

$$c_1 = -17 + 9d_1 + 3e_1 - 27K_1 \qquad c_2 = -17 + 39d_2 + 3e_2$$

$$d_1 = d_1 \qquad d_2 = d_2 \qquad (22)$$

$$e_1 = e_1 \qquad e_2 = e_2$$

$$f_1 = -18 + 6d_1 + 4e_1 - 17K_1; \quad f_2 = -18 + 39d_2 + 4e_2$$

$$g_1 = -6 + 6d_1 + e_1 - 14K_1; \quad g_2 = -6 + 14d_2 + e_2$$

$$K_1 = K_1$$

Since the coefficients in (22) are in terms of the parameters $d_1$, $e_1$, $K_1$, $d_2$, $e_2$, the constants can be determined so as to give desirable stability characteristics.

The truncation error in the predictor-corrector equations is given by

$$E_p = \frac{1}{30}(9 + 3d_1 - e_1 - 10K_1)\, h^5\, y^{(5)} \qquad (23)$$

$$E_c = \frac{1}{30}(9 - 24d_2 - e_2)\, h^5\, y^{(5)}$$

where $E_p$ and $E_c$ are the predictor and the corrector truncation error, respectively. However, the predictor and the corrector were chosen independently of fourth order. Consequently, from Henrici [11 , p 261], the truncation error of the algorithm is given by $E_c$ alone.

But, $y^{(5)}$ may not be available; thus, some means for reflecting the truncation error must be made available.

The standard technique for adjusting $P_i(M+1)$ with respect to the local truncation error is to add

$$E(M+1) = \frac{E_p}{E_c - E_p} (P_i(M) - C_i(M)) \tag{24}$$

to $P_i(M+1)$, to improve its value. Notice the fact that E(M+1) will not involve $y^{(5)}$.

Similarly, the common method of keeping track of the total truncation error is to use

$$E_T(M+1) = \frac{E_c}{E_c - E_p} (P_i(M+1) - C_i(M+1)) \tag{25}$$

as an estimate of this error at the (M+1) st step, and from (25) criteria for halving and doubling the step-size h can be generated.

According to Crane and Klopfenstein [4], the values for the constants in (22) which yield a stability situation not unlike that of Runge-Kutta methods are given by

$$\begin{aligned}
a_1 &= 1.54765200 & a_2 &= 1.0000 \\
b_1 &= -1.86750300 & b_2 &= 0. \\
c_1 &= 2.01720400 & c_2 &= 0. \\
d_1 &= -0.697353000 & d_2 &= 0.375000000 \\
e_1 &= 2.00224700 & e_2 &= 0.791666667 \\
f_1 &= -2.03169000 & f_2 &= -0.208333333 \\
g_1 &= 1.81860900 & g_2 &= 0.041666667 \\
K_1 &= -0.714320000
\end{aligned} \tag{26}$$

where the trailing zeros are used to make the method as "numerically" fourth order as possible. The coefficients in the corrector equation determine the standard Adams - Moulton corrector.

## The Generalized Predictor-Corrector

By using the standard predictor-corrector formulas as discussed in the preceding section, one inherent shortcoming arises; that of using the same fixed step-size h in integration of the various $y_i$'s. In many physical situations, and particularly in the mathematical model representing the trajectory of a ballistic rocket, the system of differential equations that arise may have some solutions varying much more rapidly than others with respect to the independent variable. Since the step-size must be chosen to give the desired accuracy in the most rapidly varying solution, by using the standard predictor-corrector, computer time may be wasted in integrating the slower varying solutions thru use of the short step-size h. Thus, the need arises for a generalized predictor-corrector algorithm in which the step-sizes may be different in each equation. To this end, we wish to use an increment $h_i$ in the ith equation where $h_{i-1} = m_i h_i$, $i = 2, 3, \ldots, N$ and $m_i$ is a positive integer.

One would like to progress to the point $x_o + h_1$ by means of one predictor-corrector step in the first equation, $m_2$ steps in the second equation, $m_2 m_3$ steps in the third equation , .., $m_2 m_3 \ldots m_N$ steps in the Nth equation. That is, we don't wish to calculate each $f_i = y_i'$ at any points between $x_o + Mh_i$ and $x_o + (M+1)h_i$, $i = 1, 2, \ldots, N$.

Suppose for the moment we have a method of calculating

$$y_i(r_i) \text{ for } r_i h_i = j h_N,\ j = 1, 2, \ldots, \frac{h_i}{h_N},\ i = 1, 2, \ldots, N-1.$$

Let this method be denoted by (*). We call (*) the generalized predictor as will be borne out in the following discussion.

To progress from $x_o$ to $x_o + h_N$ we use the following procedure:

$$\text{put } M = 1,\ r_i = \frac{h_N}{h_i},\ i = 1, 2, \ldots, N-1$$

1. predict $y_N(M)$ using standard predictor
2. predict $y_i(r_i)$ using (*), $i = 1, 2, \ldots, N-1$
3. compute $f_N(M)$
4. correct $y_N(M)$ using standard corrector
5. compute $f_N(M)$ using corrected $y_N(M)$

$$\text{put } M = M+1,\ r_i = 2r_i,\ i = 1, 2, \ldots, N-1$$

$m_N$ times

6. compute $f_{N-1}(1)$ using
   (a) $y_N(m_N)$
   (b) predicted $y_{N-1}(1)$ from standard predictor
   (c) $y_i(m_N\ r_i)$ from (*), $i = 1, 2, \ldots, N-2$
7. compute $f_{N-1}(1)$ using corrected $y_{N-1}(1)$

The same scheme is repeated to advance from $x_o + h_{N-1}$ to $x_o + 2h_{N-1}$ (using (*) advanced one step for $y_{N-1}$). Continuing in this fashion then finds the solution at $x_o + m_{N-1}\ h_{N-1} = x_o + h_{N-2}$. We are now able to compute a corrected value for $y_{N-2}$ at this point. When we finally reach $x_o + h_1$, we have evaluated $f_i$ at only $\frac{h_i}{h_N}$ points,

$i = 1, 2, \ldots, N$. For a simple example illustrating the above procedure see Appendix A.

Now it only remains to determine the generalized predictor (*) for calculating the predicted value of $y_i(\gamma)$, where $\gamma$ is the intermediate value of M, $0 < \gamma < 1$. To do this, write (20), the standard predictor, in the form.

$$P_i(\gamma) = a_1(\gamma)\, y_i(M) + b_1(\gamma)\, y_i(M-1) + c_1(\gamma)\, y_i(M-2)$$

$$+ d_1(\gamma)\, y_i(M-3) + h_i[e_1(\gamma)\, y_i'(M) + f_1(\gamma)\, y_i'(M-1) \qquad (27)$$

$$+ g_1(\gamma)\, y_i'(M-2) + K_1(\gamma)\, y_i'(M-3)]$$

where the constant coefficients are to be determined and the following notation is used:

$$P_i(\gamma) = y_i(x_o + \gamma h_i) \qquad i = 1, 2, \ldots, N$$

$$y_i(M) = y_i(x_o + Mh_i) \qquad i = 1, 2, \ldots, N \qquad (28)$$

$$y_i'(M) = y_i'(x_o + Mh_i) \qquad i = 1, 2, \ldots, N$$

By the same technique employed before, the solution set is found to be

$$a_1(\gamma) = -d_1(\gamma) - 3e_1(\gamma) + 3K_1(\gamma) + 1/4\,(\gamma^4 + 6\gamma^3 + 13\gamma^3 + 12 + r)$$

$$b_1(\gamma) = -9d_1(\gamma) + 24K_1(\gamma) + (\gamma^4 + 4\gamma^3 + 4\gamma^2)$$

$$c_1(\gamma) = 9d_1(\gamma) + 3e_1(\gamma) - 27K_1(\gamma) - 1/4\,(5\gamma^4 + 22\gamma^3 + 29\gamma^2 + 12\gamma)$$

$$d_1(\gamma) = d_1(\gamma) \qquad (29)$$

$$e_1(\gamma) = e_1(\gamma)$$

$$F_1(\gamma) = -6d_1(\gamma) + 4e_1(\gamma) - 17K_1(\gamma) - (\gamma^4 + 5\gamma^3 + 8\gamma^2 + 4\gamma)$$

$$g_1(\gamma) = 6d_1(\gamma) + e_1(\gamma) - 14K_1(\gamma) - 1/2\ (\gamma^4 + 4\gamma^3 + 5\gamma^2 + 2\gamma)$$

The corresponding truncation error in the generalized predictor is given by:

$$E = 12d_1(\gamma) - 4e_1(\gamma) + 40K_1(\gamma) + (\gamma^5 + 6\gamma^4 + 13\gamma^3 + 12\gamma^2 + 4\gamma)\ \frac{h^5 y^{(5)}}{120} \qquad (30)$$

As before, the standard technique for improving $P_i(\gamma)$ is to add

$$E(\gamma) = \frac{E\gamma}{E_c - E_1}\ (\hat{P}_i - \hat{C}_i) \qquad (31)$$

to $P_i(\gamma)$, where $E_1$ is $E_\gamma$ at $\gamma = 1$, $E_c$ is from (23), $\hat{P}_i$ is the predicted value of the previous step and $\hat{C}_i$ is the corrected value of the previous step.

Similarly, the truncation error in standard corrector is reflected in

$$E_{T_c}(M+1) = \frac{E_c}{E_c - E_p}\ (P_N(M+1) - C_N(M+1)) \qquad (32)$$

where $E_c$ and $E_p$ are the respective truncation errors in the standard predictor-corrector. By calculating $E_{T_c}(M+1)$, the validity of the answer found by using the standard predictor-corrector may be checked.

However, the task of accounting for the truncation error in the steps using the generalized predictor is somewhat different. Since only predicted values from the generalized predictor are available at intermediate steps, the truncation error in the generalized predictor is reflected in

$$E_{T_p}(\gamma) = \frac{E_1}{E_c - E_1}(\overset{\gamma}{P}_i - \overset{\gamma}{C}_i),\ i = 1, 2, \ldots, N-1 \tag{33}$$

where $E_1$ is $E_\gamma$ at $\gamma = 1$, and $\overset{\gamma}{P}_i$ and $\overset{\gamma}{C}_i$ are the predicted and corrected values, respectively, obtained at the current step. This value must be used throughout the current stage.

## Analysis of Predictor-Corrector Methods

One basic shortcoming in all predictor-corrector methods is that they require some other procedure to obtain enough values so as the method can be started. In the case of the generalized predictor-corrector, some procedure must be used to generate the first $3(m_i)$ steps in the ith equation. One method of starting the proceudre is the Runge-Kutta method.

One distinct disadvantage of the predictor-corrector methods is the difficulty in halving the step-size h. By utilizing the generalized predictor-corrector some of these difficulties can be surmounted. Still, sometimes halving one of the increments is required, necessitating interpolation or even restarting the current step.

Stability of the predictor-corrector algorithms was not discussed here due to the lengths that one is required to go to an adequate discussion. Suffice it to say, the system of differential equations to be solved must be carefully examined to determine what odd characteristics exist and then the proper type of predictor-corrector algorithm chosen to give the desired stability and accuracy. For a further discussion see [10].

Predictor-corrector methods in general, are much faster than the Runge-Kutta methods and in particular, use of the generalized predictor will reduce the function evaluations per step and hence further reduce computer time used.

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Block methods can also be applied to predictor-corrector schemes to improve their accuracy and efficiency. For a more complete discussion of block methods, see [17].

## Hybrid Method

Hybrid methods differ from those previously described in that in addition to using previous information from the M, (M-1)st, (M-2)nd, etc. steps, an outside method is used to calculate information at some (M-$\gamma$)th step, $0 < \gamma < 1$, and this information is also incorporated into the multistep procedure.

In certain cases the added information tends to stabilize the formula in which it is used and makes halving the step-size an easier task.

However, as pointed out in Gear [7], situations arise in which the use of the added information introduces much more error into the solution of the system than is desirable. For a more thorough discussion of hybrid methods, see [7] and [9].

## SUMMARY

The Runge-Kutta algorithm can be used for the solution of most systems of differential equations. Since only information from one previous step is required, the Runge-Kutta methods have favorable stability characteristics and the process of altering the step-size is an easy task. Consequently, Runge-Kutta methods are frequently used as a starting procedure for other methods such as multistep methods. However, the number of function evaluations per step required by the Runge-Kutta methods requires utilizing more computer time than multistep methods and since no measure of the truncation error is available during the calculation procedure, approximate or comparison methods are required in the program to reflect this error.

Multistep methods require fewer function evaluations per step and hence use less computer time. The truncation error can be measured effectively during the calculation procedure, thereby exhibiting the error more correctly than do Runge-Kutta methods. The stability of the predictor-corrector method should be investigated thoroughly to determine what regions of integration are required for

efficient use of these methods. Although a smaller step-size is generally required for predictor-corrector methods as compared to the Runge-Kutta method, the use of the generalized predictor-corrector method may alleviate many problems found in multistep methods and may use the least computer time of all the methods described.

The fairly attractive methods briefly described in the final section point out other methods, that stem from the Runge-Kutta and the multistep methods, that may be investigated for use in a ballistic rocket trajectory program due to their composite of the desirable characteristics of the Runge-Kutta and multistep methods.

At the present time the Atmospheric Sciences Office at White Sands Missile Range is programming the generalized predictor-corrector algorithm for use in the six-degree of freedom ballistic rocket model. It is felt that this algorithm can best accomplish the desired task with a minimum of computer time and with accuracy comparable to that of Runge-Kutta methods used currently.

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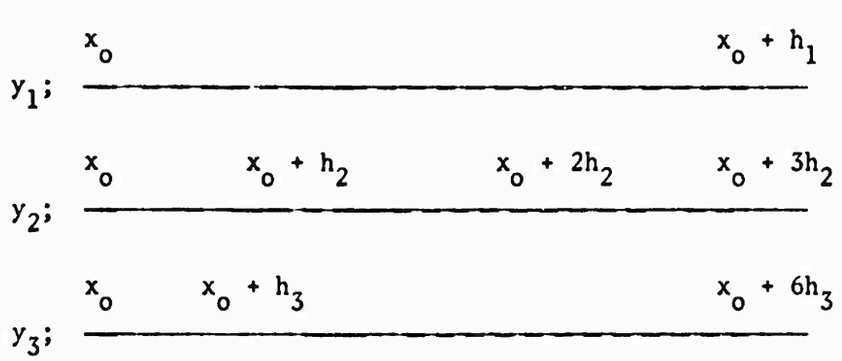
## APPENDIX A

### An Example Using the Generalized Predictor-Corrector Method

Suppose we have the following system of first order differential equations

$$\begin{aligned} y_1' &= f_1(x, y_1, y_2, y_3) \\ y_2' &= f_2(x, y_1, y_2, y_3) \\ y_3' &= f_3(x, y_1, y_2, y_3) \end{aligned} \tag{34}$$

where $y_2$ varies three times as rapidly as $y_1$ with respect to x, and $y_3$ varies twice as rapidly as $y_2$ with respect to x. We would like to find the solution of (34) at the point $x_o + h_1$. Therefore, let $h_1 = 3h_2$ and $h_2 = 2h_3$ where $h_i$ is the increment in the ith equation. This means that $m_2 = 3$ and $m_3 = 2$. The following figure will illustrate this situation.




In finding the solution of (34) at $x_o + h_1$ we will calculate $f_1$ at $x_o + h_1$ only, $f_2$ at $x_o + h_2$ and $x_o + 2h_2$ only and $f_3$ at $x_o + Kh_3$, K = 1, 2, ..., 6, and use the generalized predictor for finding the y's between these points.

The detailed procedure is the following:

1. put $r_i = \frac{h_3}{h_i}$, $i = 1, 2$

2. a. predict $y_3$ (1) using the standard predictor
   b. predict $y_1$ $(r_1) = y_1$ (1/6) using (*)
   c. predict $y_2$ $(r_2) = y_2$ (1/2) using (*)

3. compute $f_3$ (1), correct $y_3$ (1) using the standard corrector and then compute a final value of $f_3$ (1)

4. put $r_i = 2r_i = m_3 r_i$, $i = 1, 2$

5. a. predict $y_3$ (2) $= y_3$ $(m_3)$ using the standard predictor
   b. predict $y_1$ $(r_1) = y_1$ (2/6) using (*)
   c. predict $y_2$ $(r_2) = y_2$ (1) using the standard predictor

6. compute $f_3$ (2), correct $y_3$ (2) using the standard corrector, and then compute a final value of $f_3$ (2)

7. compute $f_2$ (1) using 5, correct $y_2$ (1) using the standard corrector, and then compute a final value of $f_2$ (1)

In the second equation we are now at the point $x_o + h_2$. We now advance (*) one step when using it in the second equation and it will be understood that when we say "predict $y_2$ $(r_2)$, $0<r_2<1$, using (*)" we mean we are predicting the value of $y_2$ between $x_o + h_2$ and $x_o + 2h_2$.

8. put $r_1 = 3r_1 = (m_3 + 1)\, r_1$, $r_2 = \frac{h_3}{h_2}$

9. a. predict $y_3$ (3) using the standard predictor
   b. predict $y_1$ $(r_1) = y_1$ (3/6) using (*)
   c. predict $y_2$ $(r_2) = y_2$ (1/2) using (*)

10. compute $f_3$ (3), correct $y_3$ (3) using the standard corrector and then compute a final value of $f_3$ (3)

11. put $r_1 = 4r_1 = 2m_3\, r_1$, $r_2 = 2r_2 = m_3\, r_2$

12. a. predict $y_3$ (4) = $y_3$ ($2m_3$) using the standard predictor
    b. predict $y_1$ ($r_1$) = $y_1$ (4/6) using (*)
    c. predict $y_2$ ($r_2$) = $y_2$ (1) using the standard predictor

13. compute $f_3$ (4), correct $y_3$ (4) using the standard corrector, and then compute a final value of $f_3$ (4)

14. compute $f_2$ (1) using 12, correct $y_2$ (1) using the standard corrector and then compute a final value of $f_2$ (1)

Again we advance (*) one step when using it in the second equation. We will now compute $y_2$ ($r_2$), $o<r_2<1$ between $x_o + 2h_2$ and $x_o + 3h_2$.

15. put $r_1 = 5r_1 = (2m_3 + 1)\, r_1$, $r_2 = \dfrac{h_3}{h_2}$

16. a. predict $y_3$ (5) using the standard predictor
    b. predict $y_1$ ($r_1$) = $y_1$ (5/6) using (*)
    c. predict $y_2$ ($r_2$) = $y_2$ (1/2) using (*)

17. compute $f_3$ (5), correct $y_3$ (5) using the standard corrector, and then compute a final value of $f_3$ (5)

18. put $r_1 = 6r_1 = m_3\, m_2\, r_1$, $r_2 = 2r_2 = m_3\, r_2$

19. a. predict $y_3$ (6) = $y_3$ ($m_3\, m_2$) using the standard predictor
    b. predict $y_1$ ($r_1$) = $y_1$ (1) using the standard predictor
    c. predict $y_2$ ($r_2$) = $y_2$ (1) using the standard predictor

20. from 19, compute $f_i$ (1), correct $y_i$ (1) using the standard corrector, and then compute a final value of $f_i$ (1), i = 1,2,3.

We have reached our solution at $x_0 + h_1$ by means of one standard predictor-corrector step in the first equation, $m_2 = 3$ standard predictor-corrector steps in the second equation, and $m_2 m_3 = 6$ standard predictor-corrector steps in the third equation.

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13. ABSTRACT

Numerical integration methods for solution of the system of differential equations found in ballistic rocket trajectory programs are discussed.

The general discussion entails the explicit formulas of Runge-Kutta and predictor-corrector methods and their errors, and a brief description of other methods that could be employed.

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